On the flow past a quarter infinite plate using Oseen's equations

By K. STEWARTSON
Department of Mathematics, University of Durham, Durham

AND L. HOWARTH
Department of Mathematics, University of Bristol

(Received 29 April 1959)

This paper is concerned with the determination, on the basis of Oseen's equations, of the flow past a quarter-plane with its leading edge normal to, and its side edge parallel to, a uniform incident stream. The solution is completed, except for a region in the vicinity of the corner, correct to order $\nu^{\frac{1}{2}}$ for small kinematic viscosity ν.

Away from the vicinity of the side edge the flow will approximate to the two-dimensional flow past a semi-infinite plate. This two-dimensional flow can be built up successively, if we like to think in terms of boundary conditions at the plate rather than at the edge of the boundary layer, from the potential flow associated with the uniform stream, a shear layer introduced to remove the tangential slip and a potential flow to remove the normal velocity at the plate associated with the shear layer. In the vicinity of the plate the three together give the usual boundary-layer solution.

We start our solution from this same basis, namely, the potential flow associated with the uniform stream and the shear layer to restore the no-slip condition. As a first approximation, neglecting the effects of the edges, this will be the same as for the two-dimensional problem. The normal velocity introduced by this shear layer has to be compensated by a potential flow (see § 4). This potential flow in turn (and here our problem diverges significantly from the two-dimensional problem) introduces tangential velocities with components parallel to both leading and side edges which require the introduction of a further shear layer. Over the main body of the plate this secondary shear layer is of a conventional form (§ 5) but requires special examination near the edges. In § 6 it is shown how Carrier & Lewis's (1949) solution can be modified to give the flow near the leading edge away from the tip and in § 7, the core of the paper, the flow near the side edge is determined.

In the vicinity of the side edge the extra potential flow has no component in the direction of that edge and so the solution given by Howarth (1950) for the corresponding unsteady problem is applicable. What emerges from the present calculations, however, is that Howarth's application of Rayleigh's analogy to give the excess skin friction is seriously incomplete. For, whilst this argument gives correctly the local increase of order ν in skin friction in the immediate vicinity of the side edge, it omits the widespread effects of the secondary shear layer. These are found to be of the same order in ν as the local effects.

The cross-flow in the side edge region has features of special interest. Its determination depends on a knowledge of the potential flow associated with the primary shear layer and so it depends, for instance, on the shape of the leading edge and is not, as appears to have been assumed up to now, determined completely by local conditions. This is further exemplified by the fact that it cannot be expressed in terms of what would be regarded as the natural boundary-layer variables but involves quite separately the distance from the leading edge.

---

1. Introduction

The Blasius solution for the flow past a semi-infinite flat plate has been shown to be capable of generalization to flat plates with curved leading edges provided that there are no discontinuities in the slopes of the leading edges. The flow in any plane parallel to the direction of the incident stream and containing a normal to the plate is then in fact the same as that given by the Blasius solution.

This generalization breaks down when a discontinuity of slope occurs in the leading edge for then the solution would imply discontinuities in velocity derivatives, and an extreme case is provided by a quarter infinite plate with its leading edge normal to, and its side edge parallel to, the incident stream. Though little has been published about this problem the interest it has created has been rather more perhaps than the actual problem would at first sight appear to warrant. This is because it raises problems of fundamental importance in the interplay between boundary layer and external flow and contributes to our knowledge of the role played by boundary-layer theory in obtaining solutions of the full equations of flow. One remark about the novelty of the problem is sufficient to convey the importance of it in boundary-layer theory. Calculations of the effect of the edge in increasing the skin frictional force on the plate are, as will be shown below for the Oseen equations, quite incomplete until one includes the effect of the boundary layer on the potential flow outside.

Although the essential ideas underlying the solution are simple some of the details are complicated, and all the present paper sets out to do is to attempt, on the basis of Oseen's linearization of the flow equations, to obtain as much as possible of the flow field correct to order $\nu^{\frac{1}{2}}$ for small kinematic viscosity ν. It is necessary to work to this order if the effect of the edge on the skin friction is to be estimated and it has been found possible to determine the whole of the field to this order apart from a small region near the corner. Of course with the Navier–Stokes equations as basis rather than the Oseen linearization of them the problem is more difficult, but the effects discussed here will have their counterpart there and that is the prime justification for the present paper. Some discussion of the differences to be expected with the exact equations is given at the end of the paper.

2. The Blasius problem

Some of the ideas for the present solution are contained in the two-dimensional problem of the flow past a semi-infinite plate, and as we shall require to make use of the solution of this problem anyway some discussion of it is a useful introduction. The exact solution of the Oseen equations for this problem is known;* for small ν it comprises a potential flow together with a conventional boundary layer. With U as the speed of the incident stream, the origin at the leading edge, the x-axis parallel to the incident stream and the y-axis normal to the plate, Oseen's equations may be split into the forms

$$p = -\rho U \frac{\partial \phi}{\partial x}, \quad \mathbf{v} = \operatorname{grad} \phi + \mathbf{v}', \tag{2.1}$$

$$\nabla^2 \phi = 0, \tag{2.2}$$

$$U \frac{\partial \mathbf{v}'}{\partial x} = \nu \nabla^2 \mathbf{v}', \tag{2.3}$$

$$\operatorname{div} \mathbf{v}' = 0, \tag{2.4}$$

where $\mathbf{v} = (u, v)$ is the velocity of flow. At infinity we must have $u \to U$, $v \to 0$ and at the plate $u = v = 0$.

Looking at the solution from the boundary-layer point of view in the co-ordinates x, y, the first approximation for ϕ is Ux. This would be a complete solution with $\mathbf{v}' = 0$ except that the boundary condition $u = 0$ at the plate would be violated. We remove this difficulty by introducing a shear layer outside of which $\mathbf{v}' = (u_1', v_1')$ tends to zero and in which u_1' is determined by the approximate form

$$U \frac{\partial u_1'}{\partial x} = \nu \frac{\partial^2 u_1'}{\partial y^2} \tag{2.5}$$

of the first component of (2.3), and the condition $u_1' = -U$ at the plate. This gives

$$u_1' = -U \operatorname{erfc} \eta, \quad \text{where} \quad \eta = \frac{y}{2}\left(\frac{U}{\nu x}\right)^{\frac{1}{2}}, \tag{2.6}$$

and (2.4) then implies

$$v_1' = \int_y^\infty \frac{\partial u_1'}{\partial x} dy. \tag{2.7}$$

(It is worth noting that (2.7) satisfies the shear layer form

$$U \frac{\partial v_1'}{\partial x} = \nu \frac{\partial^2 v_1'}{\partial y^2}$$

of the second component of (2.3).)

Equation (2.7) gives a non-zero value for

$$[v_1']_0 = \int_0^\infty \frac{\partial u_1'}{\partial y} dy = -U\left(\frac{\nu}{U\pi x}\right)^{\frac{1}{2}} \operatorname{sgn} y \tag{2.8}$$

on the two sides of the plate, where suffix 0 refers to values at $y = 0$. This in turn means that the boundary condition $v = 0$ must be satisfied by writing

$$\phi = Ux + \phi_1, \tag{2.9}$$

where ϕ_1 is defined by $\nabla^2 \phi_1 = 0$ and

$$\frac{\partial \phi_1}{\partial y} = -[v_1']_0 = U\left(\frac{\nu}{U\pi x}\right)^{\frac{1}{2}} \operatorname{sgn} y$$

on the two sides of the plate.

* Carrier & Lewis (1949) were the first to obtain the skin friction and a formal solution of the flow field. Kaplun (1954) and others have shown that the solution takes on a relatively simple form in parabolic co-ordinates.

One might expect that the associated value $(\partial\phi_1/\partial x)_0$ would be $O(\nu^{\frac{1}{2}})$ and different from zero; if this were so it would lead to the necessity for another shear layer type solution to remove it. In fact $(\partial\phi_1/\partial x)_0$ turns out to be zero and shear layer and potential flow are completely matched by the forms $\phi = Ux+\phi_1$ and $\mathbf{v}'$.

Two further points are worth making. First of all the conventional boundary-layer solution is $u = u_1'+U$, $v = v_1'-[v_1']_0$, and so gives $v = 0$ at the plate and $v \neq 0$ at infinity as expected. (The term $-[v_1']_0$ arises from $\partial\phi_1/\partial y$ since *inside the shear layer* $\nabla^2\phi_1 = 0$ is approximated to by $\partial^2\phi_1/\partial y^2 = 0$ which in turn implies $\partial\phi_1/\partial y = (\partial\phi_1/\partial y)_0 = -[v_1']_0$.) The velocities inside the boundary layer are then determined correct to $O(\nu^{\frac{1}{2}})$ except possibly in the vicinity of the leading edge. Secondly, in that vicinity nothing short of the full equations (2.2)–(2.4) is useful; the analysis of Carrier & Lewis (1949) solves the problem completely and shows that the skin friction determined by the boundary-layer approximation is in fact exact.

3. The quarter-plate problem: general considerations

With the origin at the corner, the x-axis parallel to the stream, the z-axis along the leading edge and the y-axis perpendicular to the plate, Oseen's equations are equivalent to

$$p = -\rho U\frac{\partial\phi}{\partial x}, \quad \mathbf{v} = \operatorname{grad}\phi+\mathbf{v}', \tag{3.1}$$

$$\nabla^2\phi = 0, \tag{3.2}$$

$$U\frac{\partial\mathbf{v}'}{\partial x} = \nu\nabla^2\mathbf{v}', \tag{3.3}$$

$$\operatorname{div}\mathbf{v}' = 0. \tag{3.4}$$

For small ν we shall now have small exceptional regions near both leading and side edges, but elsewhere on the plate we must expect a conventional type shear layer in which $\partial/\partial y \gg \partial/\partial x$ and $\partial/\partial z$.

Thus we start as before by taking $\phi = Ux$ and then removing the tangential slip by a shear layer defined by

$$U\frac{\partial u_1'}{\partial x} = \nu\frac{\partial^2 u_1'}{\partial y^2}, \tag{3.5}$$

$$\frac{\partial u_1'}{\partial x}+\frac{\partial v_1'}{\partial y} = 0, \tag{3.6}$$

$$w_1' = 0, \tag{3.7}$$

where $\mathbf{v}' = \mathbf{v}_1' = (u_1', v_1', w_1')$ and the boundary conditions are $u_1' = -U$ at the plate and $\mathbf{v}_1' \to 0$ at infinity. Equations (3.5) and (3.6) are precisely those (equations (2.4) and (2.5)) for the Blasius problem and the solution is given in (2.6) and (2.7). We may note as before that with this solution the approximate form

$$U\frac{\partial v_1'}{\partial x} = \nu\frac{\partial^2 v_1'}{\partial y^2} \tag{3.8}$$

of the second component of (3.3) is also satisfied. Again, as before, we have a velocity $[v_1']_0$ at the plate given by (2.8) for $z > 0$. We shall refer to this as the primary shear layer. We therefore require to introduce a term ϕ_1 where

$$\phi = Ux + \phi_1$$

and

$$\left.\begin{aligned}\left(\frac{\partial\phi_1}{\partial y}\right)_0 &= U\left(\frac{\nu}{U\pi x}\right)^{\frac{1}{2}} \operatorname{sgn} y \quad \text{for} \quad x > 0, z > 0 \\ &= 0 \text{ elsewhere.}\end{aligned}\right\} \tag{3.9}$$

Thus the first impact of the three-dimensional nature of the flow is felt at this stage. We now have a three-dimensional potential problem different from the corresponding two-dimensional one. Not least of these differences is, as we shall see below in §4, that $(\partial\phi_1/\partial x)_0$ and $(\partial\phi_1/\partial z)_0$ are different from zero on the plate; both are $O(\nu^{\frac{1}{2}})$. Hence to get u and w correct to order $\nu^{\frac{1}{2}}$ we have to introduce shear layers with velocity $\mathbf{v}_2' = (u_2', v_2', w_2')$ such that

$$U\frac{\partial u_2'}{\partial x} = \nu\frac{\partial^2 u_2'}{\partial y^2}, \tag{3.10}$$

$$U\frac{\partial w_2'}{\partial x} = \nu\frac{\partial^2 w_2'}{\partial y^2}, \tag{3.11}$$

$$\frac{\partial u_2'}{\partial x} + \frac{\partial v_2'}{\partial y} + \frac{\partial w_2'}{\partial z} = 0, \tag{3.12}$$

where $\mathbf{v}_2' \to 0$ at infinity and

$$u_2' = -\left(\frac{\partial\phi_1}{\partial x}\right)_0, \quad w_2' = -\left(\frac{\partial\phi_1}{\partial z}\right)_0$$

at the plate. Thus

$$v_2' = \int_y^\infty \left\{\frac{\partial u_2'}{\partial x} + \frac{\partial w_2'}{\partial z}\right\} dy \tag{3.13}$$

and again we may note that this satisfies the approximate form

$$U\frac{\partial v_2'}{\partial x} = \nu\frac{\partial^2 v_2'}{\partial y^2}, \tag{3.14}$$

for the second component.

Equation (3.13) gives a value of $[v_2']_0$ different from zero and the next step for a complete solution would be to introduce a potential ϕ_2 to restore the boundary condition at the plate by putting $(\partial\phi_2/\partial y)_0 = -[v_2']_0$ at the plate. Inside the shear layer the appropriate form of $\nabla^2\phi_2 = 0$ is $\partial^2\phi_2/\partial y^2 = 0$ and so inside the layer we should simply have to add a velocity $\partial\phi_2/\partial y = (\partial\phi_2/\partial y)_0 = -[v_2']_0$. However, v_2' and ϕ_2 are both of order ν and we need not determine them with our limited objective of obtaining the velocities correct to order $\nu^{\frac{1}{2}}$.

Thus equations (3.10) and (3.11) with their boundary conditions complete the solution correct to order $\nu^{\frac{1}{2}}$ except near the leading and side edges. We shall refer to the corresponding flow as the secondary shear layer. Outside the primary and secondary shear layers, as can be verified *a posteriori* from the detailed solutions in §§6 and 7, the error in the potential solution $\phi = Ux + \phi_1$ arising from the neglect of these edge regions in the determination of ϕ_1 is fortunately of higher order than $\nu^{\frac{1}{2}}$. Hence we can limit further attention to the shear type flows near the edges.

It is convenient in dealing with Oseen's equations in the simplest way to work with the shear layers introduced above rather than the actual boundary layers. The latter can easily be obtained from the former. Thus the primary boundary layer is obtained from the primary shear layer by superposing a velocity U in the x-direction and $-[v_1']_0$ in the y-direction. The secondary boundary layer is derived from the secondary shear layer by superposing velocities $(\partial\phi_1/\partial x)_0$ and $(\partial\phi_1/\partial z)_0$ in the x- and z-directions and $-[v_2']_0$ in the y-direction (though the latter is negligible to order ν). Similarly, in the discussions below of the edge regions the shear layer results can be translated immediately into standard boundary-layer forms.

One could say that the difference between the shear layer and boundary-layer approach lay in the difference between 'matching' at the boundary and matching at the edge of the boundary layer. In the Oseen equations to the order of accuracy of this paper the two approaches produce identical results. With the full (non-linear) boundary-layer equations standard practice is to match at the edge of the boundary layer, but it is for consideration here too whether matching at the boundary might be advantageous as suggested by Latta (1951) and Kaplun (1954) in another, though not unrelated, connexion.

Turning now to the shear layer near the leading edge and away from the vicinity of the tip, we can therein no longer neglect $\partial/\partial x$ in comparison with $\partial/\partial y$ but we can still neglect $\partial/\partial z$ and this is true *within* the layer for the potential contribution to the flow inside it. Thus the equations determining this layer with $\phi = Ux+\phi_1+\phi'$ are

$$\frac{\partial^2\phi'}{\partial x^2}+\frac{\partial^2\phi'}{\partial y^2} = 0, \tag{3.15}$$

$$U\frac{\partial \mathbf{v}'}{\partial x} = \nu\left\{\frac{\partial^2\mathbf{v}'}{\partial x^2}+\frac{\partial^2\mathbf{v}'}{\partial y^2}\right\}, \tag{3.16}$$

$$\operatorname{div}\mathbf{v}' = 0. \tag{3.17}$$

In this region, however, $w' = O(\nu^{\frac{1}{2}})$ and $\partial w'/\partial z = O(\nu^{\frac{1}{2}})$, and the effect of this term in (3.16) would be to give contributions of higher order than $O(\nu^{\frac{1}{2}})$ to u' and v', since the extent of the region in x and y is $O(\nu^{\frac{1}{2}})$. Hence we may take (3.17) in the approximate form

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y} = 0. \tag{3.18}$$

The boundary conditions are

$$u' = -U-\frac{\partial\phi_1}{\partial x}-\frac{\partial\phi'}{\partial x}, \quad v' = -\frac{\partial\phi_1}{\partial y}-\frac{\partial\phi'}{\partial y}, \quad w' = -\frac{\partial\phi_1}{\partial z}-\frac{\partial\phi'}{\partial z}$$

at the plate, and u', v', w' and $\operatorname{grad}\phi' \to 0$ at infinity. The solution is considered in §6 below.

Near the side edge and away from the vicinity of the tip we may neglect $\partial/\partial x$ in comparison with $\partial/\partial y$ but must now retain $\partial/\partial z$. Thus inside the side edge shear layer the potential equation can be written

$$\frac{\partial^2\phi'}{\partial y^2}+\frac{\partial^2\phi'}{\partial z^2} = 0, \tag{3.19}$$

where $\phi = Ux+\phi_1+\phi'$ and the equation for the shear layer velocity is

$$U\frac{\partial \mathbf{v}'}{\partial x} = \nu\left\{\frac{\partial^2 \mathbf{v}'}{\partial y^2}+\frac{\partial^2 \mathbf{v}'}{\partial z^2}\right\}. \tag{3.20}$$

In this region, since $\partial u'/\partial x$ is $O(1)$, we have to retain the full equation of continuity

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z} = 0. \tag{3.21}$$

The boundary conditions are $\mathbf{v}' \to 0$ and $\operatorname{grad}\phi' \to 0$ at infinity, whilst at the plate

$$u' = -U-\frac{\partial \phi_1}{\partial x}-\frac{\partial \phi'}{\partial x}, \quad v' = -\frac{\partial \phi_1}{\partial y}-\frac{\partial \phi'}{\partial y} \quad w' = -\frac{\partial \phi_1}{\partial z}-\frac{\partial \phi'}{\partial z}.$$

We shall find it convenient, when we consider the solution in detail in §7 below, to introduce the two-dimensional harmonic ψ conjugate to ϕ' so that we shall have

$$\frac{\partial^2 \psi}{\partial y^2}+\frac{\partial^2 \psi}{\partial z^2} = 0 \tag{3.22}$$

and boundary conditions at the plate of the form

$$u' = -U-\frac{\partial \phi_1}{\partial x}-\frac{\partial \phi'}{\partial x}, \quad v' = -\frac{\partial \phi_1}{\partial y}-\frac{\partial \psi}{\partial z}, \quad w' = -\frac{\partial \phi_1}{\partial z}+\frac{\partial \psi}{\partial y}.$$

4. The potential solution

The potential problem posed by (3.9) has the solution

$$2\pi\phi_1(x,y,z) = -\left\{\frac{U\nu}{\pi}\right\}^{\frac{1}{2}}\int_0^\infty\int_0^\infty \frac{dX\,dZ}{X^{\frac{1}{2}}\{(X-x)^2+y^2+(Z-z)^2\}^{\frac{1}{2}}}, \tag{4.1}$$

so that

$$2\pi\frac{\partial \phi_1}{\partial x} = \left\{\frac{U\nu}{\pi}\right\}^{\frac{1}{2}}\int_0^\infty\int_0^\infty \frac{(x-X)\,dX\,dZ}{X^{\frac{1}{2}}\{(X-x)^2+y^2+(Z-z)^2\}^{\frac{3}{2}}}, \tag{4.2}$$

$$2\pi\frac{\partial \phi_1}{\partial z} = \left\{\frac{U\nu}{\pi}\right\}^{\frac{1}{2}}\int_0^\infty\int_0^\infty \frac{(z-Z)\,dX\,dZ}{X^{\frac{1}{2}}\{(X-x)^2+y^2+(Z-z)^2\}^{\frac{3}{2}}}. \tag{4.3}$$

It follows that

$$2\pi\frac{\partial \phi_1}{\partial x} = \left\{\frac{U\nu}{\pi}\right\}^{\frac{1}{2}}\left\{\int_0^\infty \frac{(x-X)\,dX}{X^{\frac{1}{2}}[(x-X)^2+y^2]}+\int_0^\infty \frac{z(x-X)\,dX}{X^{\frac{1}{2}}[(x-X)^2+y^2]\,[(x-X)^2+y^2+z^2]^{\frac{1}{2}}}\right\} \tag{4.4}$$

and

$$2\pi\frac{\partial \phi_1}{\partial z} = -\left\{\frac{U\nu}{\pi}\right\}^{\frac{1}{2}}\int_0^\infty \frac{dX}{X^{\frac{1}{2}}[(X-x)^2+y^2+z^2]^{\frac{1}{2}}}. \tag{4.5}$$

Hence

$$\frac{\partial \phi_1}{\partial x} = \left\{\frac{U\nu}{\pi x}\right\}^{\frac{1}{2}} U_1(\xi,\chi), \quad \frac{\partial \phi_1}{\partial z} = \left\{\frac{U\nu}{\pi x}\right\}^{\frac{1}{2}} W_1[\surd(\xi^2+\chi^2)], \tag{4.6}$$

where

$$\chi = y/x, \quad \xi = z/x, \tag{4.7}$$

$$U_1(\xi,\chi) = -\frac{1}{2\pi}\int_0^\infty \frac{(t-1)\,dt}{t^{\frac{1}{2}}[(t-1)^2+\chi^2]}-\frac{\xi}{2\pi}\int_0^\infty \frac{(t-1)\,dt}{t^{\frac{1}{2}}[(t-1)^2+\chi^2]\,[(t-1)^2+\xi^2+\chi^2]^{\frac{1}{2}}} \tag{4.8}$$

and

$$W_1(s) = -\frac{1}{2\pi}\int_0^\infty \frac{dt}{t^{\frac{1}{2}}[(t-1)^2+s^2]^{\frac{1}{2}}}. \tag{4.9}$$

On the quarter-plane itself we have

$$U_1(\xi, 0) = -\frac{\xi}{2\pi} \lim_{\chi \to 0} \int_0^\infty \frac{(t-1)\,dt}{t^{\frac{1}{2}}[(t-1)^2+\chi^2]\,[(t-1)^2+\xi^2]^{\frac{1}{2}}}, \tag{4.10}$$

$$W_1(\xi) = -\frac{1}{2\pi} \int_0^\infty \frac{dt}{t^{\frac{1}{2}}[(t-1)^2+\xi^2]^{\frac{1}{2}}}. \tag{4.11}$$

When ξ is large

$$U_1(\xi, 0) \sim -\frac{1}{2\pi\xi^{\frac{1}{2}}} \int_0^\infty \left\{1 - \frac{1}{(1+t^4)^{\frac{1}{2}}}\right\} \frac{dt}{t^2} = \frac{0{\cdot}848}{\pi\xi^{\frac{1}{2}}} \tag{4.12}$$

and

$$W_1(\xi) \sim -\frac{1}{\pi\xi^{\frac{1}{2}}} \int_0^\infty \frac{dt}{(1+t^4)^{\frac{1}{2}}} = -\frac{1{\cdot}854}{\pi\xi^{\frac{1}{2}}}, \tag{4.13}$$

while when ξ is small

$$U_1(\xi, 0) \sim -\frac{\xi}{2\pi}\left(\log\frac{\xi}{8}+1\right) \tag{4.14}$$

and

$$W_1(\xi) \sim \frac{1}{\pi}\log\frac{\xi}{8} + O(\xi\log\xi). \tag{4.15}$$

Finally, we shall need W_1 when $y \neq 0$ but $y^2+z^2 \ll x^2$. It follows immediately from (4.6), (4.9), (4.11) and (4.15) that

$$W_1 \sim \frac{1}{2\pi}\log\frac{y^2+z^2}{64x^2}. \tag{4.16}$$

If instead of considering the quarter infinite plane $x > 0$, $y = 0$, $z > 0$ we had supposed that the leading edge was a curve of arbitrary shape but that the side edge was unaltered, the flow v_1' determined from (3.6) would be different and so would the potential problem corresponding to (3.9). In fact the logarithmic singularity in W_1 shown by (4.16) would be unchanged but it would be necessary to add to (4.16) a term $c(x)\,x^{\frac{1}{2}}$, where $c(x)$ depends on the shape of the leading edge. A corresponding addition would be necessary for U_1. In particular if the plate is the semi-infinite strip $y = 0$, $x > 0$, $R > z > 0$ where R is large, then

$$c(x) = -\frac{1{\cdot}854}{\pi R^{\frac{1}{2}}}.$$

5. The secondary shear layer

The secondary shear layer problem is posed in equations (3.10), (3.11), together with the boundary conditions

$$u_2' = U_2 = -\left\{\frac{U\nu}{\pi x}\right\}^{\frac{1}{2}} U_1(\xi, 0), \quad w_2' = W_2 = -\left\{\frac{U\nu}{\pi x}\right\}^{\frac{1}{2}} W_1(\xi) \tag{5.1}$$

at $y = 0$, $x > 0$, $z > 0$, and

$$u_2' = w_2' = 0 \quad \text{at} \quad x = 0, \quad (y > 0, z > 0).$$

We solve the problem for the whole of the quarter-plane realizing that there will be errors near the leading and side edges. These regions will be dealt with in detail below.

We see that z appears in the equations and boundary conditions entirely as a parameter and we have essentially equations to solve in two independent variables. Introducing Laplace transforms with respect to x with parameter s^2 and using Gothic type to denote transforms we have

$$\mathfrak{U}_2(s) = -\left\{\frac{U\nu}{\pi}\right\}^{\frac{1}{2}} \int_0^\infty x^{-\frac{1}{2}} U_1\left(\frac{z}{x}, 0\right) e^{-s^2 x}\, dx, \tag{5.2}$$

$$\mathfrak{W}_2(s) = -\left\{\frac{U\nu}{\pi}\right\}^{\frac{1}{2}} \int_0^\infty x^{-\frac{1}{2}} W_1\left(\frac{z}{x}\right) e^{-s^2 x}\, dx, \tag{5.3}$$

$$U s^2 \mathfrak{u}_2 = \nu \frac{\partial^2 \mathfrak{u}_2}{\partial y^2}, \quad U s^2 \mathfrak{w}_2 = \nu \frac{\partial^2 \mathfrak{w}_2}{\partial y^2}, \tag{5.4}$$

so that

$$\mathfrak{u}_2 = \mathfrak{U}_2\, e^{-(U/\nu)^{\frac{1}{2}} sy}, \quad \mathfrak{w}_2 = \mathfrak{W}_2\, e^{-(U/\nu)^{\frac{1}{2}} sy}, \tag{5.5}$$

which give the solution in operational form.

It will be sufficient for our purpose to comment on the asymptotic form of the results when ξ is either large or small. When ξ is large we find immediately that

$$u_2' = -\frac{0{\cdot}848}{\pi}\left(\frac{\nu U}{\pi z}\right)^{\frac{1}{2}} \operatorname{erfc}\eta, \quad w_2' = \frac{1{\cdot}854}{\pi}\left(\frac{\nu U}{\pi z}\right)^{\frac{1}{2}} \operatorname{erfc}\eta, \tag{5.6}$$

whilst when ξ is small

$$u_2' = \left(\frac{\nu U}{\pi x}\right)^{\frac{1}{2}} \left\{\frac{\xi}{2\pi}\left(\log\frac{\xi}{8} + 1\right)(1 - 2\eta^2)\, e^{-\eta^2} + \frac{\xi}{2\pi} A(\eta)\right\}, \tag{5.7}$$

$$w_2' = \left(\frac{\nu U}{\pi x}\right)^{\frac{1}{2}} \left\{\frac{1}{\pi} e^{-\eta^2} \log\frac{\xi}{8} + \frac{1}{\pi} B(\eta)\right\}, \tag{5.8}$$

where $A(\eta)$ and $B(\eta)$ are functions of η only, with

$$A(0) = B(0) = 0, \quad A'(0) = 4\pi^{\frac{1}{2}}, \quad B'(0) = -2\pi^{\frac{1}{2}}.$$

It will be seen that in the region on the plate where (5.6) is valid

$$\frac{\partial u_2'}{\partial y} = \frac{0{\cdot}848}{\pi^2} \frac{U}{(zx)^{\frac{1}{2}}}, \quad \frac{\partial w_2'}{\partial y} = -\frac{1{\cdot}854}{\pi^2} \frac{U}{(zx)^{\frac{1}{2}}}. \tag{5.9}$$

Therefore the contribution to the skin friction in the direction of the incident stream from the part of the plane defined by

$$0 < x < x_1, \; z_1 < z < z_2, \; z_1 \gg x_1 \text{ is } (3{\cdot}390/\pi^2)\,\mu U[x_1(z_2 - z_1)]^{\frac{1}{2}}.$$

The corresponding sideways force is $-(7{\cdot}416/\pi^2)\,\mu U[x_1(z_2 - z_1)]^{\frac{1}{2}}$. The former must be compared with the excess skin friction $\frac{1}{2}\mu U x_1$ for the rectangular strip $0 < x < x_1$, $0 < z < \infty$ found by Howarth (1950) using Rayleigh's analogy and determined by considerations in the vicinity of the edge of a type to be discussed below. In Howarth's time-variable problem of the flow engendered by a semi-infinite plane started to move parallel to its (straight) side edge the velocity is everywhere parallel to the side edge. Rayleigh's (1911) analogy as applied by Howarth to determine the flow past a quarter-plane leads to results which are seriously incomplete, since the values just quoted show that the contribution to the skin friction from the induced potential flow via the secondary shear layer are at least as important as the contribution to the skin friction from the immediate neighbourhood of the side edge. In fact the contribution from the induced potential flow tends to infinity with $|z_2 - z_1|$.

6. The flow near the leading edge excluding the corner region

Near the leading edge the flow is determined by the equations (3.15), (3.16), (3.18); again z appears only as a parameter. The solution is therefore of the type found by Carrier & Lewis (1949). Thus with parabolic co-ordinates ζ_1, ζ_2, where

$$x+iy = (\zeta_1+i\zeta_2)^2, \tag{6.1}$$

we find that ϕ' is negligible to order $\nu^{\frac{1}{2}}$ and that correct to this order

$$u' = U\left(\frac{\nu}{U\pi}\right)^{\frac{1}{2}}\frac{\zeta_2 e^{-(U\zeta_2^2/\nu)}}{\zeta_1^2+\zeta_2^2} - U\left[1+\frac{0\cdot848}{\pi}\left(\frac{\nu}{\pi Uz}\right)^{\frac{1}{2}}\right]\operatorname{erfc}\left(\frac{U}{\nu}\right)^{\frac{1}{2}}\zeta_2, \tag{6.2}$$

$$v' = -U\left(\frac{\nu}{U\pi}\right)^{\frac{1}{2}}\frac{\zeta_1 e^{-(U\zeta_2^2/\nu)}}{\zeta_1^2+\zeta_2^2}, \tag{6.3}$$

$$w' = \frac{1\cdot854}{\pi}\left(\frac{\nu}{\pi Uz}\right)^{\frac{1}{2}}\operatorname{erfc}\left(\frac{U}{\nu}\right)^{\frac{1}{2}}\zeta_2. \tag{6.4}$$

The solution correct to order $\nu^{\frac{1}{2}}$ in the vicinity of the leading edge is therefore

$$\mathbf{v} = U\mathbf{i}+\operatorname{grad}\phi_1+\mathbf{v}',$$

where ϕ_1 is given in §4, $\mathbf{v}'$ by (6.2)–(6.4), and $\mathbf{i}$ is a unit vector in the direction of the incident stream.

7. The flow near the side edge, excluding the corner region

The flow near the side edge is determined by equations (3.19)–(3.22). The first component of (3.20) can be solved independently of the others since, as can be justified *a posteriori*, $\partial\phi'/\partial x$ is negligible to order $\nu^{\frac{1}{2}}$. Furthermore, when

$$z/x \to 0, \quad \partial\phi_1/\partial x \to 0$$

for the quarter-plane whilst for a more general leading edge $\partial\phi_1/\partial x \to b(x)$, where $b(x)$ is a function of x which can be determined. We shall for simplicity confine attention to the quarter-plane though the more general solution can be obtained if desired. Hence we have to solve

$$U\frac{\partial u'}{\partial x} = \nu\left(\frac{\partial^2 u'}{\partial y^2}+\frac{\partial^2 u'}{\partial z^2}\right), \tag{7.1}$$

subject to the condition $u' = -U$ at the quarter-plane and $u' \to 0$ at infinity. This is the problem already solved by Howarth (1950) and has a solution

$$\frac{u'}{U} = -\operatorname{erfc}\eta+\frac{1}{2^{\frac{1}{2}}\pi^{\frac{3}{2}}}\int_\lambda^\infty\left[\left(\frac{v}{v-\eta}\right)^{\frac{1}{2}}-\left(\frac{v}{v+\eta}\right)^{\frac{1}{2}}\right]e^{-\frac{1}{2}v^2}K_{\frac{1}{4}}(\tfrac{1}{2}v^2)\,dv$$

when $z>0$, and

$$\frac{u'}{U} = -\frac{1}{2^{\frac{1}{2}}\pi^{\frac{3}{2}}}\int_\lambda^\infty\left[\left(\frac{v}{v-\eta}\right)^{\frac{1}{2}}+\left(\frac{v}{v+\eta}\right)^{\frac{1}{2}}\right]e^{-\frac{1}{2}v^2}K_{\frac{1}{4}}(\tfrac{1}{2}v^2)\,dv$$

when $z<0$, where as before

$$\eta = \frac{y}{2}\left(\frac{U}{\nu x}\right)^{\frac{1}{2}}, \quad \lambda = \frac{1}{2}\left[\frac{(y^2+z^2)\,U}{\nu x}\right]^{\frac{1}{2}} \tag{7.2}$$

and $K_{\frac{1}{4}}(x)$ is the Bessel function denoted in this way by Watson. The skin friction is given by

$$\tau = \mu\left[\frac{\partial u}{\partial y}\right]_0 = \mu U\left(\frac{U}{\nu x}\right)^{\frac{1}{2}}\left[\frac{1}{\pi^{\frac{1}{2}}}+\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\lambda}^{\infty} e^{-\frac{1}{4}v^2} K_{\frac{1}{4}}(\tfrac{1}{4}v^2)\frac{dv}{v}\right]. \tag{7.3}$$

This is, however, not the most convenient form for our present purpose. Let us write

$$Y = y\left(\frac{U}{\nu}\right)^{\frac{1}{2}}, \quad Z = z\left(\frac{U}{\nu}\right)^{\frac{1}{2}}, \quad u' = U\bar{u}, \quad v' = (\nu U)^{\frac{1}{2}}\bar{v},$$
$$w' = (\nu U)^{\frac{1}{2}}\bar{w}, \quad \phi_1 = \nu f, \quad \psi = \nu g, \tag{7.4}$$

and introduce the Laplace transforms with parameter s^2 with respect to x. We shall denote by $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$, $\mathfrak{f}$, $\mathfrak{g}$ the transforms of $\bar{u}, \bar{v}, \bar{w}, f, g$, respectively. Then since $\bar{u}, \bar{v}, \bar{w} = 0$ at $x = 0$,

$$s^2\mathfrak{u} = \frac{\partial^2\mathfrak{u}}{\partial Y^2}+\frac{\partial^2\mathfrak{u}}{\partial Z^2}, \tag{7.5}$$

$$s^2\mathfrak{v} = \frac{\partial^2\mathfrak{v}}{\partial Y^2}+\frac{\partial^2\mathfrak{v}}{\partial Z^2}, \tag{7.6}$$

$$s^2\mathfrak{w} = \frac{\partial^2\mathfrak{w}}{\partial Y^2}+\frac{\partial^2\mathfrak{w}}{\partial Z^2}, \tag{7.7}$$

$$s^2\mathfrak{u}+\frac{\partial\mathfrak{v}}{\partial Y}+\frac{\partial\mathfrak{w}}{\partial Z} = 0. \tag{7.8}$$

We shall solve (7.5) subject to the conditions

$$\left.\begin{aligned} \mathfrak{u} &= -\frac{1}{s^2}e^{-s\alpha Z} \quad \text{at} \quad Y = 0,\ Z > 0, \\ \frac{\partial\mathfrak{u}}{\partial Y} &= 0 \quad \text{at} \quad Y = 0,\ Z < 0, \end{aligned}\right\} \tag{7.9}$$

where $\alpha > 0$ and then we shall subsequently take the limit as $\alpha \to 0$ to obtain the solution of our problem.

To this end put

$$\mathfrak{u} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(\omega)\exp\{-is\omega Z - s(\omega^2+1)^{\frac{1}{2}}\,Y\}\,d\omega. \tag{7.10}$$

This form* satisfies (7.5) and, in view of the second condition in (7.9), $(\omega^2+1)^{\frac{1}{2}}\chi(\omega)$ is regular in the upper half-plane $\mathrm{Im}(\omega) > k_1$ for some k_1. Hence if we use suffixes + and − to denote, respectively, functions regular in the upper and lower half-planes we may put

$$\chi(\omega)\,(\omega^2+1)^{\frac{1}{2}} = \Upsilon_+(\omega). \tag{7.11}$$

However, from the first condition of (7.9)

$$\chi(\omega) = -\frac{i}{s^2(\omega+i\alpha)}+\chi_-(\omega). \tag{7.12}$$

* Since the variable χ defined in (4.7) does not appear in the remainder of the analysis no confusion arises from the introduction of the function $\chi(\omega)$, with a different meaning, in (7.10) et seq.

Therefore

$$\frac{\Upsilon_+(\omega)}{(\omega+i)^{\frac{1}{2}}}+\frac{i(-i\alpha-i)^{\frac{1}{2}}}{s^2(\omega+i\alpha)} = \chi_-(\omega)\,(\omega-i)^{\frac{1}{2}}-\frac{i}{s^2}\left[\frac{(\omega-i)^{\frac{1}{2}}-(-i\alpha-i)^{\frac{1}{2}}}{\omega+i\alpha}\right]. \tag{7.13}$$

If we assume for the moment that Υ_+ is regular for $\text{Im}\,(\omega) > -\alpha$ and $\chi_-(\omega)$ is regular for $\text{Im}\,(\omega) < 1$, then the two sides of (7.13) are equal and regular in the strip $1 > \text{Im}\,(\omega) > -\alpha$. Hence they must be regular everywhere and therefore constant. In view of the integrable singularity in the skin friction at $Y = Z = 0$, this constant is zero. Hence

$$\chi(\omega) = -\frac{i(-i\alpha-i)^{\frac{1}{2}}}{s^2(\omega+i\alpha)\,(\omega-i)^{\frac{1}{2}}}. \tag{7.14}$$

The proof is completed by noting that with this form Υ_+ and χ_- are in fact regular in the regions in which they were originally assumed to be. Equations (7.10) and (7.14) are equivalent to (7.2).

Next, to deal with $\mathfrak{v}$ and $\mathfrak{w}$ we shall write

$$\mathfrak{v} = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(\omega)\exp\{-is\omega Z-s(\omega^2+1)^{\frac{1}{2}}\,Y\}\,d\omega+\frac{\partial\mathfrak{h}}{\partial Z}, \tag{7.15}$$

$$\mathfrak{w} = \frac{1}{2\pi}\int_{-\infty}^{\infty} W(\omega)\exp\{-is\omega Z-s(\omega^2+1)^{\frac{1}{2}}\,Y\}\,d\omega-\frac{\partial\mathfrak{h}}{\partial Y}, \tag{7.16}$$

where

$$s^2\mathfrak{h} = \frac{\partial^2\mathfrak{h}}{\partial Y^2}+\frac{\partial^2\mathfrak{h}}{\partial Z^2} \tag{7.17}$$

and

$$-s(\omega^2+1)^{\frac{1}{2}}\,V(\omega)-is\omega W(\omega)+s^2\chi(\omega) = 0. \tag{7.18}$$

Equations (7.6)–(7.8) are thereby satisfied and subject to (7.18) $V(\omega)$ and $W(\omega)$ can be chosen for convenience in view of the presence of $\mathfrak{h}$ in (7.15) and (7.16).

The boundary conditions on $\mathfrak{v}$ and $\mathfrak{w}$ are

$$\mathfrak{v}+\frac{\partial\mathfrak{f}}{\partial Y}+\frac{\partial\mathfrak{g}}{\partial Z} = 0 = \mathfrak{w}+\frac{\partial\mathfrak{f}}{\partial Z}-\frac{\partial\mathfrak{g}}{\partial Y} \tag{7.19}$$

on $Y = 0, Z > 0$

and

$$\mathfrak{v}+\frac{\partial\mathfrak{f}}{\partial Y}+\frac{\partial\mathfrak{g}}{\partial Z} = 0 = \frac{\partial\mathfrak{w}}{\partial Y}+\frac{\partial^2\mathfrak{f}}{\partial Y\partial Z}-\frac{\partial^2\mathfrak{g}}{\partial Y^2} \tag{7.20}$$

on $Y = 0, Z < 0$. We may note that on $Y = 0, Z > 0$

$$\frac{\partial\mathfrak{f}}{\partial Y} = \frac{1}{s}, \tag{7.21}$$

$$\frac{\partial\mathfrak{f}}{\partial Z} = \frac{1}{\pi s}\left[\log sZ+\gamma+\tfrac{1}{2}\log\frac{s^2\nu}{4U}\right], \tag{7.22}$$

whilst on $Y = 0, Z < 0$

$$\frac{\partial\mathfrak{f}}{\partial Y} = 0, \quad \frac{\partial^2\mathfrak{f}}{\partial Y\partial Z} = 0. \tag{7.23}$$

Furthermore since the pressure p is related to $(\partial/\partial x)(\phi_1+\phi')$, we must have $\partial\mathfrak{g}/\partial Z$ and $\partial^2\mathfrak{g}/\partial Z^2$ both vanishing on $Y = 0, Z < 0$ so that we can take $\mathfrak{g} = 0$ without loss there. Hence our boundary conditions become

$$\mathfrak{v}+\frac{\partial\mathfrak{g}}{\partial Z}+\frac{1}{s} = 0, \tag{7.24}$$

$$\mathfrak{w}-\frac{\partial\mathfrak{g}}{\partial Y}+\frac{1}{\pi s}\left[\log sZ+\gamma+\tfrac{1}{2}\log\frac{s^2\nu}{4U}\right] = 0 \tag{7.25}$$

on $Y = 0$, $Z > 0$, and

$$\mathfrak{v} = 0, \quad \frac{\partial \mathfrak{w}}{\partial Y} = 0 \tag{7.26}$$

on $Y = 0$, $Z < 0$. We shall replace (7.24) by

$$\mathfrak{v} + \frac{\partial \mathfrak{g}}{\partial Z} + \frac{1}{s} e^{-s\alpha Z} = 0 \tag{7.27}$$

and ultimately let $\alpha \to 0$.

If now we choose
$$V(\omega) = -\frac{i}{s(\omega + i\alpha)} \tag{7.28}$$

so that
$$\left.\begin{aligned} \frac{1}{2\pi s}\int_{-\infty}^{\infty}\left(-\frac{i}{\omega + i\alpha}\right) e^{-is\omega Z}\, d\omega &= -\frac{1}{s} e^{-\alpha s Z} \quad (Z > 0) \\ &= 0 \quad (Z < 0) \end{aligned}\right\} \tag{7.29}$$

the boundary condition (7.27) then becomes

$$\frac{\partial}{\partial Z}(\mathfrak{g} + \mathfrak{h}) = 0 \tag{7.30}$$

for $Y = 0$, $Z > 0$.

Further, with the choice of $V(\omega)$ in (7.28) we have from (7.17)

$$W(\omega) = \frac{(\omega^2 + 1)^{\frac{1}{2}}}{s\omega(\omega + i\alpha)} - \frac{(-i\alpha - i)^{\frac{1}{2}}}{s\omega(\omega + i\alpha)(\omega - i)^{\frac{1}{2}}}, \tag{7.31}$$

where we shall imagine the contour indented to pass above the origin in the ω-plane. It follows that the contribution from W to $\partial\mathfrak{w}/\partial Y$ when $Y = 0$, $Z < 0$ is itself zero. Hence the boundary condition $\partial\mathfrak{w}/\partial Y = 0$ when $Y = 0$, $Z < 0$ is equivalent to $\partial^2\mathfrak{h}/\partial Y^2 = 0$ and since $\partial^2\mathfrak{h}/\partial Z^2$ vanishes this is equivalent to $\mathfrak{h} = 0$. Therefore the boundary conditions take on the simple form

(i) on $Y = 0$, $Z < 0$, $\mathfrak{g} = \mathfrak{h} = 0$; (7.32)

(ii) on $Y = 0$, $Z > 0$, $\mathfrak{g} + \mathfrak{h} = 0$

and
$$\left.\frac{\partial}{\partial Y}(\mathfrak{g} + \mathfrak{h}) = \frac{1}{\pi s}[\log sZ + C] + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-is\omega Z}}{\omega(\omega + i\alpha)s}\left\{(\omega^2 + 1)^{\frac{1}{2}} - \frac{(-i\alpha - i)^{\frac{1}{2}}}{(\omega - i)^{\frac{1}{2}}}\right\} d\omega,\right\} \tag{7.33}$$

where $C = \gamma + \frac{1}{2}\log s^2\nu/4U$. An equivalent form is, in the limit $\alpha \to 0$,

$$\frac{\partial}{\partial Y}(\mathfrak{g} + \mathfrak{h}) = \frac{1}{\pi s}\left[\log sZ + \frac{\pi}{2} + C\right] e^{-\alpha s Z} + \frac{1}{2\pi s}\int_{-\infty}^{\infty} \frac{e^{-is\omega Z}}{\omega(\omega + i\alpha)}\{(\omega^2 + 1)^{\frac{1}{2}} - 1\}\, d\omega. \tag{7.34}$$

It is now convenient to assume that $\mathfrak{g}$ satisfies the more general equation

$$\frac{\partial^2 \mathfrak{g}}{\partial Y^2} + \frac{\partial^2 \mathfrak{g}}{\partial Z^2} = q^2 s^2 \mathfrak{g}, \tag{7.35}$$

where $q > \alpha > 0$ and q will be made to tend to zero after α. This form then ensures that the appropriate functions are regular on the real axis and the principle of analytic continuation can then be applied.

To determine $\mathfrak{g}$, $\mathfrak{h}$ we show that a solution can be found by putting, in $Y > 0$,

$$\mathfrak{g} = -\frac{1}{2\pi s^2}\int_{-\infty}^{\infty} \Theta_+(\omega)\exp\{-s[i\omega Z + (\omega^2+q^2)^{\frac{1}{2}} Y]\}\, d\omega, \tag{7.36}$$

$$\mathfrak{h} = \frac{1}{2\pi s^2}\int_{-\infty}^{\infty} \Theta_+(\omega)\exp\{-s[i\omega Z + (\omega^2+1)^{\frac{1}{2}} Y]\}\, d\omega, \tag{7.37}$$

where $\Theta_+(\omega)$ is regular in the upper half-plane and we shall, as with (7.31), ultimately indent the contour to pass above the origin. The differential equations are thereby satisfied and so are the conditions on $Y = 0, Z < 0$, and the condition $\mathfrak{g}+\mathfrak{h} = 0$ on $Y = 0, Z > 0$. Hence we shall have a solution provided the remaining condition on $Y = 0$, $Z > 0$ is satisfied, that is, provided

$$\Theta_+(\omega)N(\omega) = -\frac{1}{\omega(\omega+i\alpha)}[(\omega^2+1)^{\frac{1}{2}}-1] - \frac{i}{\pi(\omega+i\alpha)}[C' - \log(\alpha - i\omega)] + T_-(\omega), \tag{7.38}$$

where

$$N(\omega) = (\omega^2+1)^{\frac{1}{2}} - (\omega^2+q^2)^{\frac{1}{2}}, \tag{7.39}$$

$$C' = \frac{\pi}{2} + C - \gamma = \frac{\pi}{2} + \tfrac{1}{2}\log\frac{s^2\nu}{4U} \tag{7.40}$$

and $T_-(\omega)$ is some function of ω regular in the lower half-plane.

It is shown in Appendix B that

$$\Theta_+(\omega) = -\frac{1}{\omega^2}\left[1 - \frac{1}{N_+(\omega)}\right] - \frac{iD}{\pi\omega N_+(\omega)}, \tag{7.41}$$

where $N_+(\omega)$ is determined in Appendix A, and

$$D = \frac{\pi}{2} + \tfrac{1}{2}\log\frac{s^2\nu}{16U} - 1. \tag{7.42}$$

The forms for $\mathfrak{g}$ and $\mathfrak{h}$ in (7.36) and (7.37) are thus determined. The cross-flow velocities are then obtained from (7.15) and (7.16) by adding the potential flow contributions $\partial\mathfrak{g}/\partial Z$ and $-\partial\mathfrak{g}/\partial Y$, respectively, interpreting and finally adding the flow associated with the original potential ϕ_1, obtained in §4.

The most interesting results are those giving the skin friction on the quarter-plane itself and the velocity when $y = 0, z < 0$. We shall henceforward confine our attention to these aspects of the velocity field.

8. The skin friction near the side edge

The component parallel to the main stream of the skin friction in the neighbourhood of $z = 0$ has already been determined by Howarth (equation (7.2) above); here we shall obtain its component parallel to the leading edge. We take $\alpha = 0$, $q = 0$ and note that the potential flow ϕ_1 makes no contribution to the skin friction. The contribution from $\mathfrak{g}$ to w, namely $-\partial\mathfrak{g}/\partial Y$, has to be added to (7.16) and hence the contributions to $(\partial w/\partial Y)_0$ from $\mathfrak{g}$ and $\mathfrak{h}$ amount to

$$-(\partial^2/\partial Y^2)(\mathfrak{g}+\mathfrak{h}).$$

Since $\mathfrak{g}$ satisfies Laplace's equation and $\mathfrak{h}$ satisfies (7.17) and $(\mathfrak{g}+\mathfrak{h})$ vanishes on the plate (and therefore so does $(\partial^2/\partial Z^2)(\mathfrak{g}+\mathfrak{h})$) the contributions from $\mathfrak{g}$ and $\mathfrak{h}$ amount to $s^2\mathfrak{h}$, simply. Hence

$$\left(\frac{\partial w}{\partial Y}\right)_{Y=0+} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is\omega Z}\left\{\Theta_+(\omega)+\frac{\omega^2+1}{\omega^2}-\frac{(-i)^{\frac{1}{2}}}{\omega^2}(\omega+i)^{\frac{1}{2}}\right\}d\omega, \qquad (8.1)$$

where, as explained previously, the contour of integration is the real axis, indented at the origin into the upper half-plane. If $Z < 0$ the integral vanishes. When $Z > 0$, the contribution from the last two terms of (8·1) is

$$-\frac{1}{2}+\frac{1}{\pi}\int_1^{\infty}\frac{(t-1)^{\frac{1}{2}}}{t^2}e^{-stZ}\,dt. \qquad (8.2)$$

Using (7.41) and (A.12), the first term in (8.1) can be written as

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is\omega Z}\left[-\frac{1}{\omega^2}\left\{1-\frac{1}{N_+(\omega)}+\frac{i\omega}{\pi}\log\left(-\frac{i\omega}{2e}\right)\right\}-\frac{iD}{\pi\omega N_+(\omega)}+\frac{i}{\pi\omega}\log\left(-\frac{i\omega}{2e}\right)\right]d\omega \qquad (8.3)$$

of which the last term integrates to

$$\frac{1}{\pi}\left(\gamma+\log\frac{sZ}{2e}\right). \qquad (8.4)$$

Hence

$$\left(\frac{\partial \mathfrak{w}}{\partial Y}\right)_{Y=0+} = -\frac{1}{2}+\frac{1}{\pi}\left(\gamma+\log\frac{sZ}{2e}\right)+\frac{D}{\pi}$$
$$+\frac{1}{\pi^2}\int_0^{\infty}dt\,e^{-stZ}\left\{\frac{tD-\pi}{tN_-(-it)}+\frac{\pi}{t}\right\}+\frac{1}{\pi}\int_1^{\infty}\frac{e^{-stZ}\,dt}{t^2}\left[\frac{(tD-\pi)(t^2-1)^{\frac{1}{2}}}{\pi N_-(-it)}+(t-1)^{\frac{1}{2}}\right]. \qquad (8.5)$$

In inverting (8.5) to get $(\partial w/\partial y)_{y=0+}$ it is convenient to note that the inverse transformation of

$$e^{-stZ} \quad \text{is} \quad \frac{\zeta t}{x\pi^{\frac{1}{2}}}e^{-\zeta^2t^2}, \quad \text{where} \quad \zeta=\frac{Z}{2x^{\frac{1}{2}}}=\frac{z}{2}\left(\frac{U}{\nu x}\right)^{\frac{1}{2}}, \qquad (8.6)$$

and of

$$\log\frac{16U}{s^2\nu}e^{-stZ} \quad \text{is} \quad \frac{\zeta t}{x\pi^{\frac{1}{2}}}e^{-\zeta^2t^2}\left[\log\frac{16Ux}{\nu}+g(\zeta t)\right], \qquad (8.7)$$

where

$$g(\tau)=\frac{2}{\pi^{\frac{1}{2}}}\int_0^{\infty}e^{-v^2}dv\left\{\frac{1}{v^2+\tau^2}-\log(v^2+\tau^2)\right\}.$$

Using these results it then follows that

$$\frac{\pi x}{U\mu}\left[\mu\frac{\partial w}{\partial y}\right]_{y=0+,z>0} = -\frac{1}{2}+\frac{\zeta}{\pi^{\frac{1}{2}}}\int_0^{\infty}t\,e^{-\zeta^2t^2}Q(t,\zeta,x)\,dt$$
$$+\frac{\zeta}{\pi^{\frac{1}{2}}}\int_1^{\infty}e^{-\zeta^2t^2}\left\{\frac{(t-1)^{\frac{1}{2}}}{t}+(t^2-1)^{\frac{1}{2}}Q(t,\zeta,x)\right\}dt, \qquad (8.8)$$

where

$$2\pi N_-(-it)\,Q(t,\zeta,x) = -\log\frac{16Ux}{\nu}-g(\zeta t)-2+\pi-\frac{2\pi}{t}. \qquad (8.9)$$

It is worth noting that this component of skin friction is of much more complicated form than that for the component parallel to the main stream. The complication arises principally from the presence of the term $\log(16Ux/\nu)$ in (8.9) and means, for example, that the cross-flow does not possess the similarity properties of the main stream component.

In particular, when ζ is large

$$\mu\left[\frac{\partial w}{\partial y}\right]_{y=0+,z>0} = -\frac{\mu U}{\pi x}\left\{1-\frac{1}{2\pi^{\frac{3}{2}}\zeta}\left(\log\frac{z}{8x}+\frac{\pi}{2}\right)+O\left(\frac{\log\zeta}{\zeta}\right)^2\right\}, \tag{8.10}$$

in conformity with (5.8).

When ζ is small

$$\mu\left[\frac{\partial w}{\partial y}\right]_{y=0+,z>0} = -\frac{1}{2\pi^{\frac{1}{2}}(-\frac{3}{4})!}\left(\frac{\mu U}{\pi x\zeta^{\frac{1}{2}}}\right)\left[\log\frac{16Ux}{\nu}+2-\pi-\frac{\{(-\frac{3}{4})!\}'}{(-\frac{3}{4})!}\right]. \tag{8.11}$$

9. Flow in the plane $y = 0$, $z < 0$

In a similar way the velocity distribution in $y = 0$, $z < 0$ may be found. It can be verified that when ζ is large

$$w \to \left(\frac{U\nu}{\pi^3 x}\right)^{\frac{1}{2}}\log\left|\frac{z}{8x}\right| \tag{9.1}$$

in conformity with (4.15), whilst near $\zeta = 0$ we find

$$w \sim -\frac{2}{(-\frac{3}{4})!}\left(\frac{U\nu\zeta}{\pi^3 x}\right)^{\frac{1}{2}}\left[\log\frac{16Ux}{\nu}+2-\pi-\frac{\{(-\frac{3}{4})!\}'}{(-\frac{3}{4})!}\right].$$

10. Discussion

In the previous sections, expressions correct to order $\nu^{\frac{1}{2}}$ when ν is small have been obtained on the basis of Oseen's equations for the velocity distribution at all points in the fluid except those near the corner of the plate.

The $O(1)$ effect arises, as might be expected from a Blasius-type boundary layer—the primary boundary layer—and produces a frictional stress of order $\nu^{\frac{1}{2}}$ in the direction of the incident stream. What emerges from our calculations is that the modification to this flow arising from the side edge is not confined to the vicinity of that edge. Continuity considerations demand an outflow from the primary boundary layer and the potential flow thereby engendered, which is $O(\nu^{\frac{1}{2}})$, gives rise to a secondary boundary layer in which there are velocity components parallel to each edge. These components give rise to stresses of order ν parallel to each edge.

The primary and secondary boundary layers together with the potential flow give velocities correct to order $\nu^{\frac{1}{2}}$ except for local regions near both edges. The flow near the leading edge can be obtained by the same method (and virtually the same analysis) as that employed by Carrier & Lewis (1949) for the two-dimensional problem and doesn't call for any special comment here.

In the side-edge region the flow component in the direction of the main stream is determined by the same equations as Howarth solved for the time-variable flow engendered by a semi-infinite plane with a straight edge started to move parallel to the edge. It gives rise to an additional stress of order ν in the direction of the main stream. Hence, in addition to the primary stress, the stress in the direction of the main stream is made up of two quite separate components, the one a direct edge effect local to the edge, the other arising from the secondary boundary layer and significant over much of the plate. Both are of order ν. Howarth's application of Rayleigh's argument to his time-variable problem as a means of obtaining information about the steady problem of the quarter-plane discussed in this paper is therefore incomplete. His argument gives quite correctly the local effect of the edge, but does not take into consideration the widespread effect arising from the secondary boundary layer.

The cross-flow components in the side-edge zone have two features of interest. First of all they depend for their determination, through appropriate boundary conditions, on the potential flow originating from the primary boundary layer and in particular on its behaviour near the side edge. (The component of this potential flow in the direction of the main flow vanishes as the edge is approached and so does not influence the velocity component in the direction of the main flow in this zone.) An important consequence is that if the quarter infinite plate is replaced by another lamina also having $y = z = 0$ as a side edge but otherwise with a different perimeter, the cross-flow in the side-edge zone will also be different. Secondly, in this zone the component of the flow in the direction of the main flow is a function of

$$\eta = \frac{y}{2}\left(\frac{U}{\nu x}\right)^{\frac{1}{2}}, \quad \zeta = \frac{z}{2}\left(\frac{U}{\nu x}\right)^{\frac{1}{2}}$$

only, but the cross-flow components v, w are such that $x^{\frac{1}{2}}v$ and $x^{\frac{1}{2}}w$, in addition to depending on η and ζ, depend also on $\log Ux/\nu$ and this fact considerably complicates the solution.

Of course the use of Oseen's equations simplifies the problem very appreciably and it is interesting to speculate on the effects to be found with the Navier–Stokes equations. The non-linear character of these latter equations implies a much more complicated coupling between the various equations than is the case in our analysis. However, it is possible to draw certain conclusions. We shall, as here, have a primary boundary layer and the outflow from it will engender a potential flow which will be identical, apart from a numerical factor, with that in § 4. This will give rise to a secondary boundary layer of a form similar to § 5, except that now there is some coupling between the components in the u and w directions through the equation of continuity. The difference is most marked at small values of z/x.

However, the greatest divergences from our analysis are to be expected in the side-edge zone. The coupling there is particularly strong, all three components being interdependent. Thus in particular the component in the direction of the main flow, which in the Oseen formulation was obtained independently of the cross-flow, can no longer be obtained without reference to the cross-flow and therefore to the potential flow. Since, as has already been pointed out, the potential flows for laminae having the same side edge but different leading edges differ in the vicinity of the side edge, the behaviour in the side-edge zone of the component in the direction of the main flow can be longer therefore be considered as a purely local effect. In addition to its dependence on η and ζ a direct dependence on x is also to be expected.

A consideration of the form of the solution near the side edge also shows that this region cannot be $O(\nu^{\frac{1}{2}})$ in thickness. For if it were, while the viscous terms would be $O(1)$, the inertia terms of the Navier–Stokes equation would be $O(\log \nu^{-1})$, since w is $O(\nu^{\frac{1}{2}} \log \nu^{-1})$ near $y = z = 0$. In the limit $\nu \to 0$ the viscous terms would then be negligible in comparison with the inertia terms and so could not balance them as is required. It is hoped to consider this point in a later paper.

Appendix A

The determination of $N_-(\omega)$ and $N_+(\omega)$

In the strip $-q < \mathrm{Im}\,\omega < q$,

$$N(\omega) = (\omega^2 + 1)^{\frac{1}{2}} - (\omega^2 + q^2)^{\frac{1}{2}} \tag{A.1}$$

is regular and non-vanishing. Hence using Cauchy's theorem we may write $N(\omega) = N_+(\omega)/N_-(\omega)$ where N_+ and N_- are regular in the upper and lower half-planes respectively and are given by (see, for example, Carrier & Di Prima (1956))

$$\frac{N'(\omega)}{N(\omega)} = \frac{N'_+(\omega)}{N_+(\omega)} - \frac{N'_-(\omega)}{N_-(\omega)} = -\frac{1}{2\pi i}\int_C \frac{\xi\, d\xi}{(\xi^2+1)^{\frac{1}{2}}(\xi^2+q^2)^{\frac{1}{2}}(\xi-\omega)}, \tag{A.2}$$

where C consists of two lines C_1, C_2 parallel to the real axis lying in $-q < \mathrm{Im}\,\xi < q$ and enclosing ω. Thus we may write

$$\frac{N'_-(\omega)}{N_-(\omega)} = -\frac{1}{2\pi i}\int_{C_2} \frac{\xi\, d\xi}{(\xi^2+1)^{\frac{1}{2}}(\xi^2+q^2)^{\frac{1}{2}}(\xi-\omega)} \tag{A.3}$$

taking C_2 to be above C_1 and the integration to be in the direction $\mathrm{Re}\,\xi$ increasing since both sides are regular in $\mathrm{Im}\,\omega < q$. Similarly

$$\frac{N'_+(\omega)}{N_+(\omega)} = -\frac{1}{2\pi i}\int_{C_1} \frac{\xi\, d\xi}{(\xi^2+1)^{\frac{1}{2}}(\xi^2+q^2)^{\frac{1}{2}}(\xi-\omega)}. \tag{A.4}$$

The contour C_2 may now be deformed into the two sides of the straight line joining $\xi = iq$ to $\xi = i$, whence

$$\frac{N'_-(\omega)}{N_-(\omega)} = -\frac{1}{\pi i}\int_q^1 \frac{t\, dt}{(t+i\omega)(1-t^2)^{\frac{1}{2}}(t^2-q^2)^{\frac{1}{2}}}. \tag{A.5}$$

The properties of $N_-(\omega)$ follow by integration on choosing $N_-(0) = 1$, so that $N_+(0) = 1 - q$.

The behaviour of $N_-(\omega)$ near $\omega = 0$ if q is small but not zero can be obtained by noting that from A. 5

$$\frac{N'_-(0)}{N_-(0)} = -\frac{1}{\pi i}\int_q^1 \frac{dt}{(1-t^2)^{\frac{1}{2}}(t^2-q^2)^{\frac{1}{2}}} = -\frac{1}{\pi i}\log\frac{4}{q} + O(q). \tag{A.6}$$

Hence we have for ω small

$$N_-(\omega) = 1 + \frac{i\omega}{\pi}\log\frac{4}{q} \tag{A.7}$$

approximately, if $0 \leqslant \omega \ll q \ll 1$, with a corresponding expression for $N_+(\omega)$.

Next we shall need the behaviour of $N_-(\omega)$ in the limit $q \to 0$. Then

$$\frac{N'_-(\omega)}{N_-(\omega)} = -\frac{1}{\pi i}\int_0^1 \frac{dt}{(t+i\omega)(1-t^2)^{\frac{1}{2}}}. \tag{A.8}$$

Hence
$$\log N_-(\omega) = \frac{1}{\pi i}\int_0^\omega \frac{1}{\sqrt{(1+\omega^2)}}\log\left[-i\left\{\frac{\sqrt{(1+\omega^2)}-1}{\omega}\right\}\right]. \tag{A. 9}$$

Thus, when ω is small
$$\log N_-(\omega) = \frac{1}{\pi i}\int_0^\omega \log\left(-\frac{i\omega}{2}\right)d\omega$$
$$= \frac{\omega}{\pi i}\log\left(-\frac{\omega i}{2e}\right),$$

so that
$$N_-(\omega) = 1-\frac{\omega i}{\pi}\log\left(-\frac{\omega i}{2e}\right) \tag{A. 10}$$

when ω is small.

When ω is large it is convenient to integrate A. 8 under the integral sign to give
$$\log N_-(\omega) = +\frac{1}{\pi}\int_0^1 \frac{\log(t+i\omega)-\log t}{(1-t^2)^{\frac{1}{2}}}dt$$
$$= \tfrac{1}{2}\log i\omega - \frac{1}{\pi}\int_0^{\pi/2}\log\sin\theta\, d\theta + \frac{1}{\pi i\omega} + O\left(\frac{1}{\omega^2}\right) \tag{A. 11}$$

so that
$$N_-(\omega) = (2i\omega)^{\frac{1}{2}} + \frac{1}{\pi}\left(\frac{2}{i\omega}\right)^{\frac{1}{2}} + O(\omega^{-\frac{3}{2}}) \tag{A. 12}$$

when ω is large.

The corresponding results for $N_+(\omega)$ follow immediately from (A. 10) and (A. 12).

Appendix B

The determination of $\Theta_+(\omega)$

The purpose of this appendix is to apply the Wiener–Hopf technique to the determination of $\Theta_+(\omega)$ in equation (7.38).

We can write this equation, with the help of Appendix A, in the form
$$\Theta_+(\omega)\,N_+(\omega) = \sum_{r=1}^{9} J_r, \tag{B. 1}$$

where
$$J_1 = -\left(\frac{N_+(\omega)-N_+(0)}{\omega(\omega+i\alpha)}\right), \tag{B. 2}$$
$$J_2 = \left(\frac{1-q}{i\alpha}\right)\left(\frac{N_-(\omega)-N_-(0)}{\omega}\right), \tag{B. 3}$$
$$J_3 = -\left(\frac{1-q}{i\alpha}\right)\left(\frac{N_-(\omega)-N_-(-i\alpha)}{\omega+i\alpha}\right), \tag{B. 4}$$
$$J_4 = \left(\frac{1-q}{i\alpha}\right)\left(\frac{N_-(0)-N_-(-i\alpha)}{\omega+i\alpha}\right), \tag{B. 5}$$
$$J_5 = -\frac{iN_-(-i\alpha)}{\pi(\omega+i\alpha)}(C'-\log(\alpha-i\omega)), \tag{B. 6}$$
$$J_6 = -\left(\frac{N_-(-i\alpha)}{\omega(\omega+i\alpha)}\right)[(\omega^2+q^2)^{\frac{1}{2}}-q], \tag{B. 7}$$
$$J_7 = -\frac{iC'}{\pi(\omega+i\alpha)}[N_-(\omega)-N_-(-i\alpha)], \tag{B. 8}$$
$$J_8 = \frac{i}{\pi}\left[\frac{N_-(\omega)-N_-(-i\alpha)}{\omega+i\alpha}\right]\left[\log(\alpha-i\omega)+\frac{i\pi}{\omega}[(\omega^2+q^2)^{\frac{1}{2}}-q]\right], \tag{B. 9}$$
$$J_9 = T_-(\omega)\,N_-(\omega). \tag{B. 10}$$

We notice at once that J_1, J_4, J_5 are regular in the upper half-plane $\mathrm{Im}\,\omega > -\alpha$ and J_2, J_3, J_7 are regular in the lower half-plane $\mathrm{Im}\,\omega < q$. Moreover, J_6 and J_8 can be split into parts J_{6+} and J_{8+} regular in the upper half-plane and J_{6-} and J_{8-} regular in the lower half-plane. Hence, if we assume that $\Theta_+(\omega)\,N_+(\omega)$ is regular in $\mathrm{Im}\,\omega > -\alpha$ and $T_-(\omega)\,N_-(\omega)$ is regular in $\mathrm{Im}\,\omega < q$ we can rewrite (B. 1) in the form

$$\Theta_+(\omega)\,N_+(\omega)-J_1-J_4-J_5-J_{6+}-J_{8+} = J_2+J_3+J_{6-}+J_7+J_{8-}+J_9$$

in which the two sides are regular in different overlapping halves of the ω-plane. Hence, both must be regular everywhere and constant; we can complete the proof by noting that with these forms $\Theta_+(\omega)\,N_+(\omega)$ and $T_-(\omega)\,N_-(\omega)$ are in fact regular in the regions in which they were assumed to be. From a consideration of the properties of $\Theta_+(\omega)$ when ω is large it follows that the constant is zero.

Hence we have

$$\Theta_+(\omega)\,N_+(\omega) = J_1+J_4+J_5+J_{6+}+J_{8+}. \tag{B. 11}$$

For $\omega \neq 0$, we can now conveniently take the limit $\alpha \to 0$ though we shall retain q as a small parameter. Then we have immediately

$$J_1 = -\frac{N_+(\omega)-N_+(0)}{\omega^2}, \tag{B. 12}$$

$$J_4 = \frac{(1-q)}{i\pi\omega}\left\{\log\frac{q}{4}+O(q)\right\}, \tag{B. 13}$$

$$J_5 = -\frac{iN_-(0)}{\pi\omega}\,(C'-\log(-i\omega)). \tag{B. 14}$$

To calculate J_{6+} and J_{8+} we must use the same techniques as in Appendix A. Thus, since

$$J_6 = -\frac{N_-(0)}{\omega^2}\,[(\omega^2+q^2)^{\frac{1}{2}}-q], \tag{B. 15}$$

$$J_{6+} = -\frac{N_-(0)}{2\pi i}\int_{C_1}\frac{d\zeta}{\zeta^2}\,\frac{[(\zeta^2+q^2)^{\frac{1}{2}}-q]}{\zeta-\omega}, \tag{B. 16}$$

where C_1 is a contour starting at $-\infty$, ending at $+\infty$ and crossing the imaginary axis in the interval $(-iq, 0)$. This contour can be deformed into the sides of the negative imaginary axis from infinity to $-iq$ where it is indented.

Hence
$$J_{6+} = -\frac{N_-(0)}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\,d\theta}{(q-i\omega\cos\theta)} \tag{B. 17}$$

$$= +\frac{N_-(0)}{\pi}\left\{\frac{i}{\omega^2}\sqrt{(q^2+\omega^2)}\log\frac{\omega-\sqrt{(\omega^2+q^2)}}{iq}-\frac{1}{i\omega}+\frac{q\pi}{2\omega^2}\right\}. \tag{B. 18}$$

Thus
$$J_{6+} = \frac{N_-(0)}{\pi i\omega}\left\{-1+\log\frac{2}{q}+\log(-i\omega)+O(q)\right\} \tag{B. 19}$$

when $q \ll \omega$.

Next,
$$J_{8+} = \frac{i}{\pi}\int_{C_1}\frac{\{N_-(\zeta)-N_-(0)\}}{\zeta(\zeta-\omega)}\left\{\log(-i\zeta)+\frac{i\pi}{\zeta}[(\zeta^2+q^2)^{\frac{1}{2}}-q]\right\}d\zeta \tag{B. 20}$$

and deforming the contour into the two sides of the negative axis indented at $-iq$ we find that

$$\pi J_{8+} = I_1 + I_2,$$

$$I_1 = \int_0^q \left[\frac{N_-(-it)-N_-(0)}{t(t-i\omega)}\right] dt,$$

$$I_2 = \int_q^\infty \left[\frac{N_-(-it)-N_-(0)}{t(t-i\omega)}\right]\left[1-\frac{(t^2-q^2)^{\frac{1}{2}}}{t}\right] dt.$$

Now in I_2 we can split the range of integration into (q,ϵ), (ϵ,∞) where $q \ll \epsilon \ll 1$. Then in the range (ϵ,∞)

$$\int_\epsilon^\infty \left|\frac{N_-(it)-N_-(0)}{t(t-i\omega)}\right|\left|1-\frac{(t^2-q^2)^{\frac{1}{2}}}{t}\right| dt < \int_\epsilon^\infty \left|\frac{N_-(it)-N_-(0)}{t^3(t-i\omega)}\right| q^2\,dt = O(q^2).$$

The contribution to I_2 from the remaining range is equal to

$$\int_1^{\epsilon/q} \frac{[N_-(iqT)-N_-(0)]}{(qT-i\omega)\,T}\left[1-\frac{(T^2-1)^{\frac{1}{2}}}{T}\right] dT = \int_1^{\epsilon/q} \frac{(qT/\pi)\log(4/q)}{T(qT-i\omega)}\left[1-\frac{(T^2-1)^{\frac{1}{2}}}{T}\right] dT$$

in view of the restrictions on ϵ.

Also
$$I_1 = \int_0^1 \left[\frac{N_-(iqT)-N_-(0)}{(qT-i\omega)\,T}\right] dT = \int_0^1 \frac{(qT/\pi)\log(4/q)}{T(qT-i\omega)}\, dT.$$

Therefore when $q \ll \omega$

$$J_{8+} = O(q\log q). \tag{B.21}$$

The results of Appendix A together with equations (B. 11), (B. 12), (B. 13), (B. 14), (B. 18) and (B. 21) then serve to determine $\Theta_+(\omega)$. In the limit $\alpha \to 0$, $q \to 0$ equations (B. 11), (B. 12), (B. 13), (B. 19) and (B. 21) then show that

$$\Theta_+(\omega) = -\frac{1}{\omega^2}\left[1-\frac{1}{N_+(\omega)}\right] - \frac{iD}{\pi\omega N_+(\omega)}, \tag{B.22}$$

where $N_+(\omega)$ is as determined in Appendix A with $q = 0$ and $D = C' - 1 - \log 2$.

As it stands the behaviour of (B. 22) near $\omega = 0$ is apparently a source of difficulty. However, this may be avoided by noting that since $\Theta_+(\omega)$ is regular in the upper half-plane, $\operatorname{Im}\omega > 0$, the contours in (7.36), (7.37) and those dependent on them may, as already anticipated, be indented to pass above the origin and the integrations performed in standard fashion.

REFERENCES

Carrier, G. F. 1946 *Quart. J. Appl. Math.* **4**, 367–70.

Carrier, G. F. & Di Prima, R. C. 1956 *J. Math. Phys.* **35**, 359–83.

Carrier, G. F. & Lewis, J. A. 1949 *Quart. J. Appl. Math.* **7**, 228–33.

Howarth, L. 1950 *Proc. Camb. Phil. Soc.* **46**, 127–40.

Kaplun, S. 1954 *J. Appl. Math. Phys.* **5**, 111–35.

Latta, G. E. 1951 Thesis, Calif. Inst. of Technology.

Rayleigh, Lord 1911 *Phil. Mag.* (6) **21**, 697–711.

Sowerby, L. 1958 *Quart. J. Mech. Appl. Math.* **11**, 290–301.